MODULATIONAL INSTABILITY AND ROGUE WAVES ACCORDING TO NONLINEAR SCHRODINGER EQUATION

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Abstract:*In the following paper we present the numerical simulations of the Nonlinear Schrodinger Equation (NLSE), according to which it is possible to observe nonlinear wave dynamics. Our main research concerns to two basic phenomena: Modulational Instability (MI) and Rogue Waves (RW). It's been a while since these effects have been investigated in various branches of physics, for example in astrophysics, plasma physics, also MI and RW might be observed in the atmosphere of the earth.*

Key words: *Modulational Instability, Rogue Waves, Nonlinear Schrodinger Equation.*

1. Introduction

It is well known, that the Nonlinear Schrodinger Equation (NLSE) has an exact stationary single-soliton solution. Our work can be separated in two basic parts. At first we apply the NLSE and derive the Benjamin-Feir-Lighthill criterion [1] for modulational instability; for this purpose we use the perturbative method to proceed the stability analysis; the second part concerns Rogue Waves. One of the solutions of the NLSE is the rational solution that could describe the rogue wave propagation [2]. The importance of the rogue wave in the study of the ocean waves are due to the fact that the amplitude of the rogue wave can reach more than twice the value of the surrounding chaotic waves (Bludov et al. 2010). Actually, its appearance cannot be predicted, so this wave represents a real danger on the ships and boats and thus many studies deal with the ocean rogue wave (Janssen 2003; Khaarif et al. 2009; Osborne 2009; Osborne et al. 2009). Understanding the origin of the rogue wave appearance is currently a matter of debate (Ruban et al. 2010). Nonlinear wave studies related to the rogue wave phenomena can also be found in optics [3], superfluid-He4 [4] and optical cavities [5]. Whereas theoretical models can now fairly well describe oceanic rogue waves[6] there is, as far as we know, no corresponding theory for the atmospheric rogue waves that have been observed [7]. We shall therefore here point out a way to describe such atmospheric waves.

2. Modulational instability

We apply the NLSE in the following form:

$$A_t = i(\beta_2/2)A_{xx} + \gamma i|A|^2 A\,, \tag{1.1}$$

where A_t and A_{xx} are first order time and second order space derivatives respectively. Now the posing of the problem is as follows: let's discuss the spatially uniform solution of the (1.1) equation and find its instability condition; on the other hand, it means the assumption:

$$A_{xx} = 0.$$

As a result, (1.1) equation becomes:

$$A_t = \gamma i|A|^2 A\,, \tag{1.2}$$

now to apply the linear approximation, let's make the following substitution in the (2.2) equation:

$$A_t = \gamma i|a|^2 A\,, \tag{1.3}$$

where $a = const$ is in general the complex amplitude. We can find the solution of the (2.3) equation in the

$$A = ae^{\gamma i|a|^2 t}. \tag{1.4}$$

form. Now our goal is to investigate the stability of the solution of the (1.4) equation, for this purpose we use the perturbative method, it means, that (1.4) must be perturbed by some perturbative term with the amplitude $\varepsilon \ll 1$. So we can rewrite (1.4) in the form:

$$u = ae^{\gamma i|a|^2 t} + \varepsilon ae^{\gamma i|a|^2 t} = ae^{\gamma i|a|^2 t}(1+\varepsilon). \tag{1.5}$$

Here $\varepsilon = \varepsilon(x,t) \ll 1$ and it is in general a complex function. After substituting (1.5) in the (1.1) and make some mathematical transformations we shall get the expression for the time derivative of $\varepsilon(x,t)$:

$$\frac{\partial \varepsilon}{\partial t} = i\frac{\partial^2 \varepsilon}{\partial x^2} + \gamma i|a|^2(\varepsilon + \bar{\varepsilon}). \tag{1.6}$$

Now for the space array with the periodicity of $2L$ let's find the solution of (1.6) in the following form:

$$\varepsilon = c(t)e^{i\frac{x}{L}\pi} + d(t)e^{-i\frac{x}{L}\pi}, \tag{1.7}$$

where $c(t)$ and $d(t)$ are in general time depended complex amplitudes. After substituting (1.7) in the (1.6) we shall get the equation for eigenfunctions and eigenvalues:

$$\begin{pmatrix} \frac{dc}{dt} \\ \frac{d\bar{d}}{dt} \end{pmatrix} = i\begin{pmatrix} A & B \\ -B & -A \end{pmatrix}\begin{pmatrix} c \\ \bar{d} \end{pmatrix}, \tag{1.8}$$

where

$$A = -\left(\frac{\pi}{L}\right)^2 + \gamma|a|^2, \qquad B = \gamma|a|^2 c,$$

Here $\bar{d}$ is the complex conjugate of d. After solving the (1.8) system we shall have

$$\lambda = \frac{\pi}{L}\sqrt{2\gamma|a|^2 - \left(\frac{\pi}{L}\right)^2}, \qquad 2\gamma|a|^2 > \left(\frac{\pi}{L}\right)^2. \tag{1.9}$$

Where λ is the eigenvalue. Finally for the instability condition we get

$$L > \frac{\pi}{|a|\sqrt{2\gamma}}, \tag{1.10}$$

which is known as the Benjamine-Feir-Lighthill criterion.

2. Rogue waves

We use the NLSE in the following form:

$$i\frac{\partial A}{\partial x} - \frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2} + \gamma|A|^2 A = 0. \tag{2.1}$$

Analytical solution [8]:

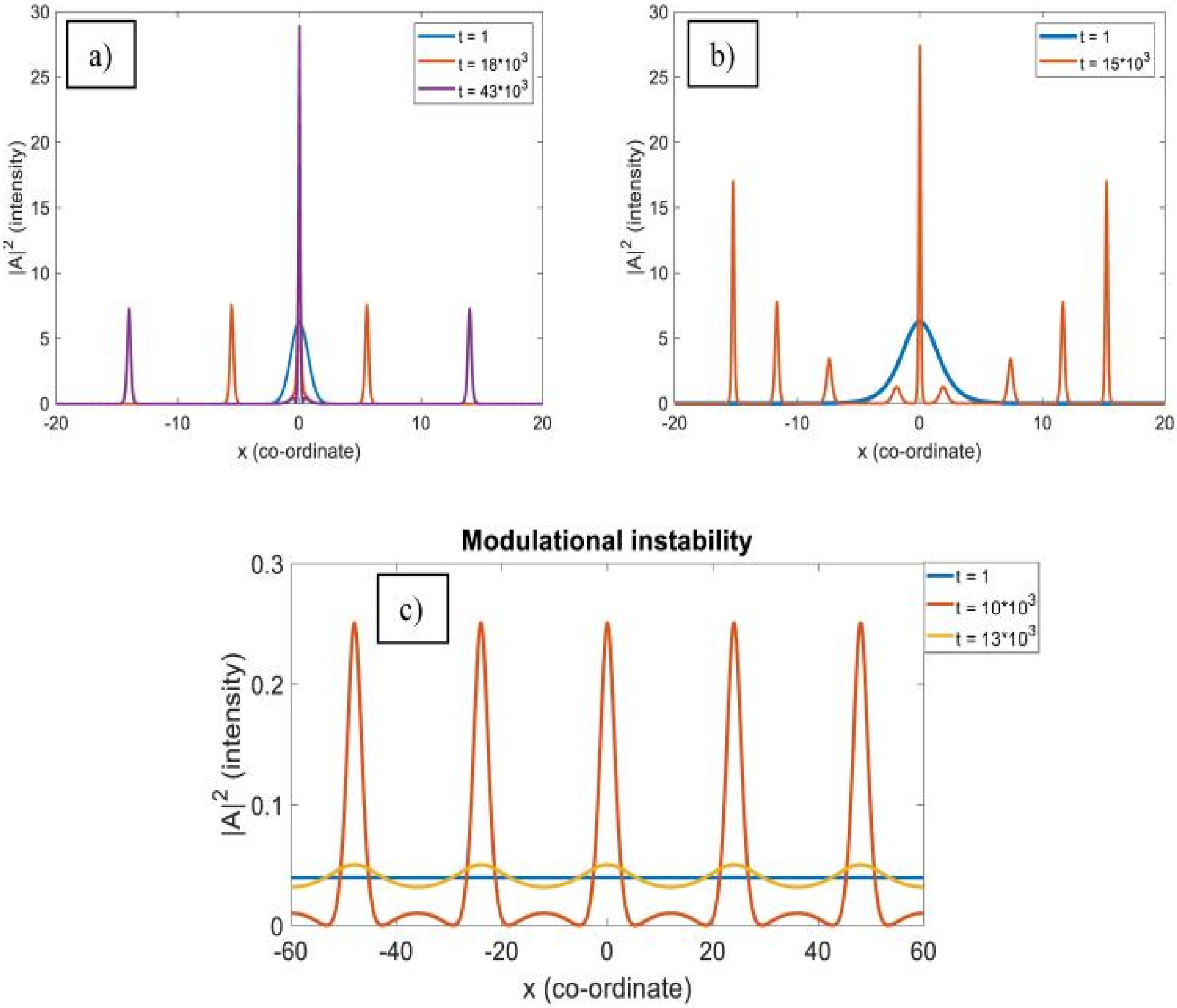
$$A_j(x,t) = \left[(-1)^j + \frac{G_j(x,t) + ixH_j(x,t)}{D_j(x,t)}\right]\exp(ix). \tag{2.2}$$

For atmospheric waves $G_1 = 4,\ H_1 = 8,\ D_1 = 1 + 4t^2 + 4x^2$. We shall have:

$$A_1(x,t) = \left[1 - 4\frac{1 + 2ix}{1 + 4t^2 + 4x^2}\right]\exp(ix) \tag{2.3}$$

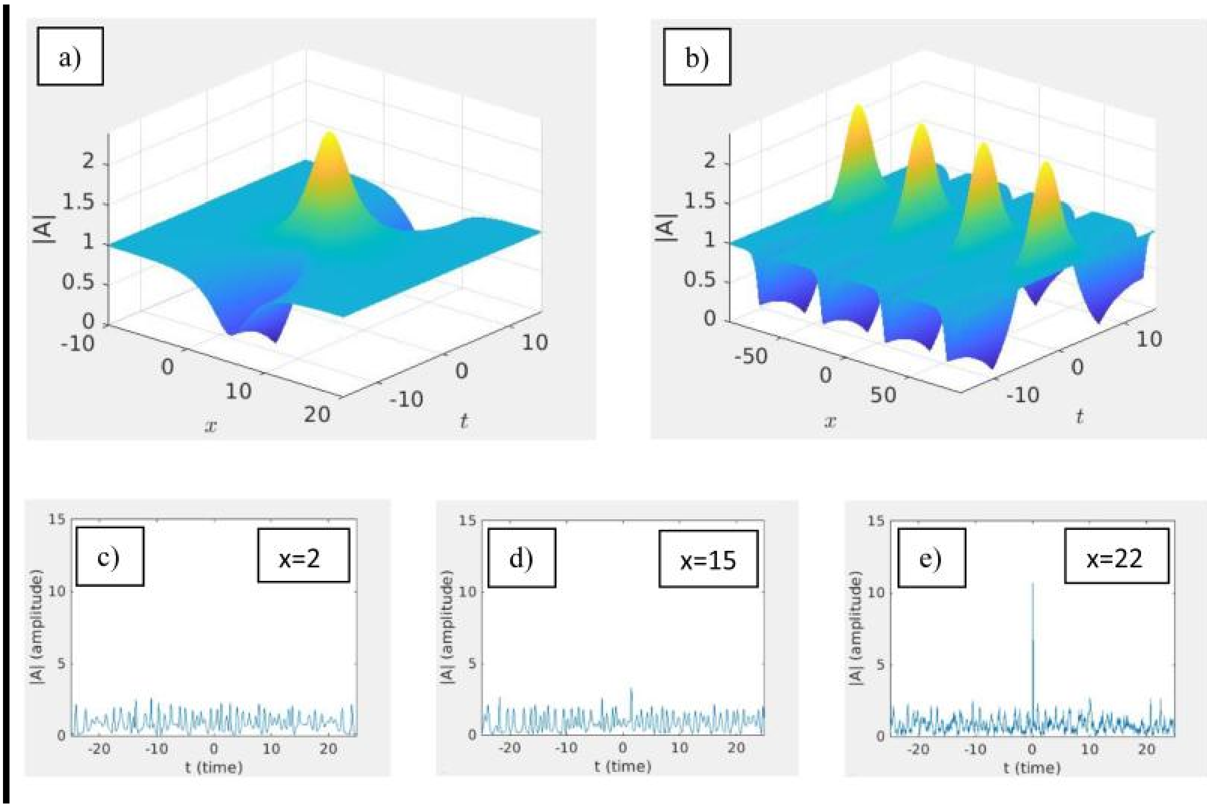
3. Numerical results

3.1 Modulational instability



3.1: a), b): MI for two different waves; $L = 2, \beta_2 = -2, \gamma = 2$.
a) $A(x,0) = 2.5 * \exp(-x^2/L)$, *b)* $A(x,0) = 2.5 * \text{sech}(x/L)$.
c) $A(x,0) = a(1 + \varepsilon\cos(\pi x/L))$; $\beta_2 = 2$; $\gamma = 2$; $L = 12$.

3.2 Rogue waves



3.2: a), b): RW; $\beta_2 = 25, \gamma = 0.5$.
c), d), e): RW; $\beta_2 = 0.21, \gamma = 3.01$.

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